

The higher-derivative example from section 2.4 of hep-th/0701238

The identity which we will prove here comes from appendix A of hep-th/0111128. The declaration of the indices, Weyl tensor and covariant derivative:

```
{i,j,m,n,k,p,q,l,r,r#}::Indices(vector).
C_{m n p q}::WeylTensor.
\nabla{#}::Derivative.
\nabla_{r}{ C_{m n p q} }::SatisfiesBianchi.
```

Assigning list property Indices to $\{i, j, m, n, k, p, q, l, r, r\# \}$.

Assigning property WeylTensor to C_{mnpq} .

Assigning property Derivative to $\nabla\#$.

Assigning property SatisfiesBianchi to $\nabla_r C_{mnpq}$.

The identity which we want to prove:

```
Eij:=- C_{i m k l} C_{j p k q} C_{l p m q} + 1/4 C_{i m k l} C_{j m p q} C_{k l p q}
      - 1/2 C_{i k j l} C_{k m p q} C_{l m p q}:
```

```
E:= C_{j m n k} C_{m p q n} C_{p j k q} + 1/2 C_{j k m n} C_{p q m n} C_{j k p q}:
```

```
exp:= \nabla_{i}{\nabla_{j}{ @ (Eij) }} - 1/6 \nabla_{i}{\nabla_{i}{ @ (E) }};
```

$$\begin{aligned} exp := & \nabla_i \nabla_j (-C_{imkl} C_{jpkq} C_{lpmq} + \frac{1}{4} C_{imkl} C_{jmpq} C_{klpq} - \frac{1}{2} C_{ikjl} C_{kmpq} C_{lmpq}) \\ & - \frac{1}{6} \nabla_i \nabla_i (C_{jmnk} C_{mpqn} C_{pjkq} + \frac{1}{2} C_{jkmn} C_{pqmn} C_{jkpq}); \end{aligned}$$

First apply the product rule to write out the derivatives,

```
@distribute!(%): @prodrule!(%):
```

```
@distribute!(%): @prodrule!(%):
```

```
@prodsort!(%): @canonicalise!(%): @rename_dummies!(%): @sumflatten!(%):
```

```
@collect_terms!(%);
```

$$\begin{aligned}
exp := & C_{ijmn}C_{ikmp}\nabla_q\nabla_jC_{nkpq} - C_{ijmn}\nabla_kC_{ipmq}\nabla_pC_{jqnk} - 2C_{ijmn}\nabla_iC_{mkpq}\nabla_pC_{jknq} \\
& - C_{ijmn}\nabla_kC_{ikmp}\nabla_qC_{jpnq} + C_{ijmn}C_{ikmp}\nabla_j\nabla_qC_{nkpq} - 2C_{ijmn}\nabla_iC_{jkmp}\nabla_qC_{nkpq} \\
& - C_{ijmn}C_{ikpq}\nabla_m\nabla_pC_{jqnk} - \frac{1}{4}C_{ijmn}C_{ijkp}\nabla_q\nabla_mC_{nqkp} + \frac{1}{4}C_{ijmn}\nabla_kC_{ijpq}\nabla_pC_{mnkq} \\
& - \frac{1}{2}C_{ijmn}\nabla_iC_{jkpq}\nabla_kC_{mnpq} - \frac{1}{4}C_{ijmn}\nabla_kC_{ijkp}\nabla_qC_{mnpq} - \frac{1}{4}C_{ijmn}C_{ijkp}\nabla_m\nabla_qC_{nqkp} \\
& - \frac{1}{2}C_{ijmn}\nabla_iC_{mnkp}\nabla_qC_{jqkp} + \frac{1}{4}C_{ijmn}C_{ikpq}\nabla_j\nabla_kC_{mnpq} \\
& - \frac{1}{2}C_{ijmn}C_{ijmk}\nabla_p\nabla_qC_{npkq} + C_{ijmn}\nabla_kC_{ijmp}\nabla_qC_{nqkp} - C_{ijmn}\nabla_kC_{ijmp}\nabla_qC_{nkpq} \\
& + \frac{1}{2}C_{ijmn}C_{ikpq}\nabla_m\nabla_jC_{nkpq} + \frac{1}{2}C_{ijmn}\nabla_iC_{mkpq}\nabla_nC_{jkpq} - \frac{1}{2}C_{ijmn}\nabla_iC_{jkpq}\nabla_mC_{nkpq} \\
& + \frac{1}{2}C_{ijmn}C_{ikpq}\nabla_j\nabla_mC_{nkpq} + \frac{1}{2}C_{ijmn}C_{ikmp}\nabla_q\nabla_qC_{jknq} + C_{ijmn}\nabla_kC_{ipmq}\nabla_kC_{jpnq} \\
& - \frac{1}{4}C_{ijmn}C_{ijkp}\nabla_q\nabla_qC_{mnkp} - \frac{1}{2}C_{ijmn}\nabla_kC_{ijpq}\nabla_kC_{mnpq};
\end{aligned}$$

Because the identity which we intend to prove is only supposed to hold on Einstein spaces, we set the divergence of the Weyl tensor to zero,

`@substitute!(%)(\nabla_{i}{C_{k i l m}} -> 0, \nabla_{i}{C_{k m l i}} -> 0);`

$$\begin{aligned}
exp := & C_{ijmn}C_{ikmp}\nabla_q\nabla_jC_{nkpq} - C_{ijmn}\nabla_kC_{ipmq}\nabla_pC_{jqnk} - 2C_{ijmn}\nabla_iC_{mkpq}\nabla_pC_{jknq} \\
& - C_{ijmn}C_{ikpq}\nabla_m\nabla_pC_{jqnk} - \frac{1}{4}C_{ijmn}C_{ijkp}\nabla_q\nabla_mC_{nqkp} + \frac{1}{4}C_{ijmn}\nabla_kC_{ijpq}\nabla_pC_{mnkq} \\
& - \frac{1}{2}C_{ijmn}\nabla_iC_{jkpq}\nabla_kC_{mnpq} + \frac{1}{4}C_{ijmn}C_{ikpq}\nabla_j\nabla_kC_{mnpq} \\
& + \frac{1}{2}C_{ijmn}C_{ikpq}\nabla_m\nabla_jC_{nkpq} + \frac{1}{2}C_{ijmn}\nabla_iC_{mkpq}\nabla_nC_{jkpq} - \frac{1}{2}C_{ijmn}\nabla_iC_{jkpq}\nabla_mC_{nkpq} \\
& + \frac{1}{2}C_{ijmn}C_{ikpq}\nabla_j\nabla_mC_{nkpq} + \frac{1}{2}C_{ijmn}C_{ikmp}\nabla_q\nabla_qC_{jknq} + C_{ijmn}\nabla_kC_{ipmq}\nabla_kC_{jpnq} \\
& - \frac{1}{4}C_{ijmn}C_{ijkp}\nabla_q\nabla_qC_{mnkp} - \frac{1}{2}C_{ijmn}\nabla_kC_{ijpq}\nabla_kC_{mnpq};
\end{aligned}$$

This expression should vanish upon use of the Bianchi identity. By expanding all tensors using their Young projectors, this becomes manifest,

`@young_project_product!(%):`

`@sumflatten!(%):`

`@canonicalise!(%):`

`@rename_dummies!(%):`

`@collect_terms!(%);`

`@canonicalise:` not applicable.

$$exp := 0;$$

This proves the identity.